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Investigation of anti-plane shear behavior of two collinear cracks in the piezoelectric materials by using the non-local theory

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Abstract

In this paper, the interaction between two collinear cracks in the piezoelectric materials under anti-plane shear loading was investigated by using the non-local theory for impermeable crack-face conditions. By using the Fourier transform, the problem can be solved with the help of two pairs of triple integral equations. The solutions are obtained by using the Schmidt method. Numerical examples are provided to show the effect of the geometry of the interacting cracks. Contrary to the previous results, it is found that no stress and electric displacement singularity is present near the crack tip.

Keywords: Non-local theory; Piezoelectric materials; Cracks; Fourier transform

1. Introduction

It is well known that piezoelectric materials produce an electric field when deformed, and undergo deformation when subjected to an electric field. The coupling nature of piezoelectric materials has attracted wide applications in electric-mechanical and electric devices, such as electric-mechanical actuators, sensors and structures. When subjected to mechanical and electrical loads in service, these piezoelectric materials can fail prematurely due to their brittleness and presence of defects or flaws produced during their manufacturing process. Therefore, it is important to study the electro-elastic interaction and fracture behaviors of piezoelectric materials.

Many studies have been made on the electro-elastic fracture mechanics based on the modeling and analyzing of one crack in the piezoelectric materials. (see, for examples, Deeg (1980), Pak (1990, 1992), Sosa (1992), Suo et al. (1992), Park and Sun (1995a,b), Zhang and Tong (1996), Zhang et al. (1998), Gao et al. (1997), Wang (1992) and Shindo et al. (1996)). The problem of the interacting fields among multiple cracks

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in a piezoelectric materials has been studied by Han and Wang (1999). In Han's paper, the crack is treated as a continuous distributed dislocations with the density function to be determined according to the conditions of external loads and crack surface. Most recently, Chen and Karihaloo (1999) considered an infinite piezoelectric ceramic with impermeable crack-face boundary condition under arbitrary electro-mechanical impact. Sosa and Khutoryansky (1999) investigated the response of piezoelectric bodies created by internal electric sources. For the sake of analytical simplification, the assumption that the crack surfaces are impermeable to electric fields was widely used in the works (Pak, 1990, 1992; Suo et al., 1992; Suo, 1993; Park and Sun, 1995a,b; Chen and Karihaloo, 1999). However, these solutions contain stress and electric displacement singularity. This is not reasonable according to the physical nature. To overcome the stress singularity in the classical elastic theory, Eringen et al. (1977) and Eringen (1978, 1979) used the non-local theory to study the state of stress near the tip of a sharp line crack in an elastic plate subjected to the uniform tension, shear and anti-plane shear. These solutions did not contain any stress singularity, thus resolved a fundamental problem that persisted over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way.

In the present paper, the interaction between two collinear symmetrical cracks subjected to the anti-plane shear in piezoelectric materials was investigated by using the non-local theory for impermeable crack-face conditions. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects. Fourier transform technique is applied and a mixed boundary-value problem is reduced to two pair of triple integral equations. In solving the triple integral equations, the crack surface displacement and electric potential are expanded in a series using Jacobi's polynomials, and the Schmidt method (Morse and Feshbach, 1958) is used to obtain the solution. This process is quite different from those adopted in the references mentioned above. As expected, the solution in this paper does not contain the stress and electric displacement singularity at the crack tip, thus clearly indicating the physical nature of the problem, namely, in the vicinity of the geometrical discontinuities in the body, the non-local intermolecular forces are dominant. For such problems, therefore, one can resort to theories incorporating non-local effects, at least in the neighborhood of the discontinuities.

2. Basic equations of non-local piezoelectric materials

According to non-local theory (see e.g. Eringen (1979)), for the anti-plane shear problem, the basic equations of linear, homogeneous, isotropic, piezoelectric materials, with vanishing body force are

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \tag{2}$$

$$\tau_{kz}(X) = \int_V [c'_{44}(|X' - X|) w_{,k}(X') + e'_{15}(|X' - X|)\phi_{,k}(X')]\,\mathrm{d}V(X'), \quad k = x, y \tag{3}$$

$$D_k(X) = \int_V [e'_{15}(|X' - X|) w_{,k}(X') - \varepsilon'_{11}(|X' - X|)\phi_{,k}(X')]\,\mathrm{d}V(X'), \quad k = x, y \tag{4}$$

where the only difference from classical theory is Eqs. (3) and (4), in which the stress $\tau_{zk}(X)$ and the electric displacement $D_k(X)$ at a point X depends on $w_{,k}(X)$ and $\phi_{,k}(X)$, at all points of the body. w and ϕ are the mechanical displacement and electric potential. For homogeneous and isotropic piezoelectric materials there exist only three material parameters, $c'_{44}(|X' - X|)$, $e'_{15}(|X' - X|)$ and $\varepsilon'_{11}(|X' - X|)$ which are functions of the distance $|X' - X|$. The integrals in (3) and (4) are over the volume V of the body enclosed within a surface ∂V.

As discussed in the papers (see e.g. Eringen (1974, 1977)), it can be assumed in the form of $c'_{44}(|X' - X|)$, $e'_{15}(|X' - X|)$ and $\varepsilon'_{11}(|X' - X|)$ for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves, the non-local moduli c'_{44}, e'_{15} and ε'_{11} are assumed to have the same form as:

$$(c'_{44}, e'_{15}, \varepsilon'_{11}) = (c_{44}, e_{15}, \varepsilon_{11})\alpha(|X' - X|) \tag{5}$$

$$\alpha(|X' - X|) = \alpha_0 \exp[-(\beta/a)^2(X' - X)(X' - X)] \tag{6}$$

where β is a constant, a is the lattice parameter and c_{44}, e_{15}, ε_{11} are the shear modulus, piezoelectric coefficient and dielectric parameter, respectively. α_0 is determined by the normalization condition:

$$\int_V \alpha(|X' - X|)\,\mathrm{d}V(X') = 1 \tag{7}$$

In the present work, the non-local moduli was given by (5) and (6). Substituting (6) into (7), it can be obtained, in two dimensional space,

$$\alpha_0 = \frac{1}{\pi}(\beta/a)^2 \tag{8}$$

Substitution of Eqs. (5) and (6) into Eqs. (3) and (4) yields

$$\tau_{kz}(X) = \int_V \alpha(|X' - X|)\sigma_{kz}(X')\,\mathrm{d}V(X'), \quad k = x, y \tag{9}$$

$$D_k(X) = \int_V \alpha(|X' - X|)D^c_k(X')\,\mathrm{d}V(X'), \quad k = x, y \tag{10}$$

where

$$\sigma_{kz} = c_{44}w_{,k} + e_{15}\phi_{,k}, \quad k = x, y \tag{11}$$

$$D^c_k = e_{15}w_{,k} - \varepsilon_{11}\phi_{,k}, \quad k = x, y \tag{12}$$

The expressions (11) and (12) are the classical constitutive equations.

3. The crack model

Consider an infinite piezoelectric body containing two collinear symmetric impermeable cracks of length $1 - b$ along the x-axis with the distance between two cracks being $2b$ (see Fig. 1). The piezoelectric boundary-value problem is considerably simplified if we consider only the out-of-plane displacement and the in-plane electric fields. The plate is subjected to a constant stress $\tau_{yz} = -\tau_0$, and a constant electric displacement $D_y = -D_0$ along the surface of the cracks. As discussed in the references by Narita and Shindo (1998); Shindo et al. (1996); Yu and Chen (1998) and Eringen (1979), the boundary conditions of the present problem are:

$$\tau_{yz}(x, 0) = -\tau_0, \quad b \leqslant |x| \leqslant 1 \tag{13}$$

$$D_y(x, 0) = -D_0, \quad b \leqslant |x| \leqslant 1 \tag{14}$$

$$w(x, 0) = \phi(x, 0) = 0, \quad |x| < b, \ \ |x| > 1 \tag{15}$$

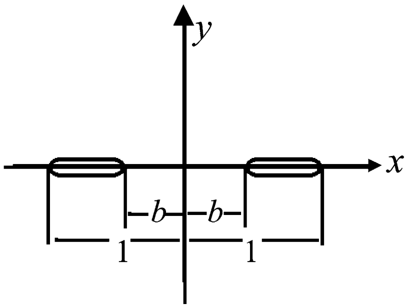

Fig. 1. Cracks in a piezoelectric materials body under anti-plane shear.

$$w(x,y) = \phi(x,y) = 0, \quad \text{for } (x^2+y^2)^{1/2} \to \infty \tag{16}$$

Substituting Eqs. (9) and (10) into Eqs. (1) and (2), respectively, using Green–Gauss theorem, it can be obtained (see e.g. Eringen (1979)):

$$\int_V \int \alpha(|x'-x|,|y'-y|)[c_{44}\nabla^2 w(x',y') + e_{15}\nabla^2\phi(x',y')]\,\mathrm{d}x'\,\mathrm{d}y' - \left(\int_{-1}^{-b} + \int_b^1\right)\alpha(|x'-x|,0)\{\sigma_{yz}(x',0)\}\,\mathrm{d}x' = 0 \tag{17}$$

$$\int_V \int \alpha(|x'-x|,|y'-y|)[e_{15}\nabla^2 w(x',y') - \varepsilon_{11}\nabla^2\phi(x',y')]\,\mathrm{d}x'\,\mathrm{d}y' - \left(\int_{-1}^{-b} + \int_b^1\right)\alpha(|x'-x|,0)\{D_y^c(x',0)\}\,\mathrm{d}x' = 0 \tag{18}$$

where the curly bracket indicates a jump at the crack line, i.e. $\{\sigma_{yz}(x,0)\} = \sigma_{yz}(x,0^+) - \sigma_{yz}(x,0^-)$, $\{D_y^c(x,0)\} = D_y^c(x,0^+) - D_y^c(x,0^-)$. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \leqslant x \leqslant \infty$, $0 \leqslant y \leqslant \infty$ only. Under the applied anti-plane shear load on the unopened surfaces of the crack, the displacement field and the electric displacement possess the following symmetry regulations

$$w(x,-y) = -w(x,y), \qquad \phi(x,-y) = -\phi(x,y) \tag{19}$$

Using Eq. (19), we find that

$$\{\sigma_{yz}(x,0)\} = 0 \tag{20}$$

$$\{D_y^c(x,0)\} = 0 \tag{21}$$

Hence the line integrals in (17) and (18) vanish. By taking the Fourier transform of (17) and (18) with respect to x', it can be shown that the general solutions of (17) and (18) are identical to that of

$$c_{44}\left[\frac{\mathrm{d}^2\bar{w}(s,y)}{\mathrm{d}y^2} - s^2\bar{w}(s,y)\right] + e_{15}\left[\frac{\mathrm{d}^2\bar{\phi}(s,y)}{\mathrm{d}y^2} - s^2\bar{\phi}(s,y)\right] = 0 \tag{22}$$

$$e_{15}\left[\frac{\mathrm{d}^2\bar{w}(s,y)}{\mathrm{d}y^2} - s^2\bar{w}(s,y)\right] - \varepsilon_{11}\left[\frac{\mathrm{d}^2\bar{\phi}(s,y)}{\mathrm{d}y^2} - s^2\bar{\phi}(s,y)\right] = 0 \tag{23}$$

almost everywhere. Here a superposed bar indicates the Fourier transform,

$$\bar{f}(s,y) = \int_{-\infty}^{\infty} f(x,y)\exp(\mathrm{i}sx)\,\mathrm{d}x$$

The general solutions of Eqs. (22) and (23) ($y \geqslant 0$) satisfying (16) are, respectively:

$$w(x,y) = \frac{2}{\pi}\int_0^{\infty} A(s)\mathrm{e}^{-sy}\cos(xs)\,\mathrm{d}s, \qquad \phi(x,y) - \frac{e_{15}}{\varepsilon_{11}}w(x,y) = \frac{2}{\pi}\int_0^{\infty} B(s)\mathrm{e}^{-sy}\cos(xs)\,\mathrm{d}s \tag{24}$$

where $A(s)$, $B(s)$ are unknown functions to be determined by the boundary conditions.

The stress field and the electric displacement, according to (9) and (10), are given by, respectively

$$\tau_{yz}(x,y) = \frac{2}{\pi}\int_0^{\infty}[-\mu sA(s) - e_{15}sB(s)]\,\mathrm{d}s\int_0^{\infty}\mathrm{d}y' \times \int_{-\infty}^{\infty}[\alpha(|x'-x|,|y'-y|) + \alpha(|x'-x|,|y'+y|)]\mathrm{e}^{-sy'}\cos(sx)\,\mathrm{d}x \tag{25}$$

$$D_y(x,y) = \frac{2}{\pi}\int_0^{\infty}\varepsilon_{11}sB(s)\,\mathrm{d}s\int_0^{\infty}\mathrm{d}y'\int_{-\infty}^{\infty}[\alpha(|x'-x|,|y'-y|) + \alpha(|x-x|,|y+y|)]\mathrm{e}^{-sy'}\cos(sx)\,\mathrm{d}x \tag{26}$$

where $\mu = c_{44} + (e_{15}^2/\varepsilon_{11})$.

Substituting for α from (6), according to the reference (see e.g. Eringen (1979)) and the boundary conditions (13)–(15), it can be obtained

$$\frac{2}{\pi}\int_0^{\infty} sA(s)\mathrm{erfc}(\varepsilon s)\cos(sx)\,\mathrm{d}s = \frac{1}{\mu}\left(\tau_0 + \frac{e_{15}D_0}{\varepsilon_{11}}\right), \quad b \leqslant |x| \leqslant 1 \tag{27}$$

$$\frac{2}{\pi}\int_0^{\infty} A(s)\cos(sx)\,\mathrm{d}s = 0, \quad |x| < b, \ |x| > 1 \tag{28}$$

and

$$\frac{2}{\pi}\int_0^{\infty} sB(s)\mathrm{erfc}(\varepsilon s)\cos(sx)\,\mathrm{d}s = -\frac{D_0}{\varepsilon_{11}}, \quad b \leqslant |x| \leqslant 1 \tag{29}$$

$$\frac{2}{\pi}\int_0^{\infty} B(s)\cos(sx)\,\mathrm{d}s = 0, \quad |x| < b, \ |x| > 1 \tag{30}$$

where $\varepsilon = a/2\beta$, $\mathrm{erfc}(z) = 1 - \Phi(z)$, $\Phi(z) = (2/\sqrt{\pi})\int_0^z \exp(-t^2)\,\mathrm{d}t$.

Since the only difference between the classical and the non-local equations is in the introduction of the function $\mathrm{erfc}(\varepsilon s)$, it is logical to utilize the classical solution to convert the system (27)–(30) to an integral equation of the second kind which is generally better behaved. For $a = 0$, $\mathrm{erfc}(\varepsilon s) = 1$ and Eqs. (27)–(30) reduce to the triple integral equations for same problem in classical piezoelectric materials. To determine the unknown functions $A(s)$ and $B(s)$, the dual-integral equations (27)–(30) must be solved.

4. Solution of the triple integral equation

The triple integral equations (27)–(30) cannot be transformed into the second Fredholm integral equation (Eringen, 1979), because the kernel of the second kind Fredholm integral equation in the paper of Eringen (Eringen, 1979) is divergent. The kernel of the second kind Fredholm integral equation in Eringen's paper (Eringen, 1979) can be written as follows:

$$L(x,u) = (xu)^{1/2}\int_0^{\infty} tk(\varepsilon t)J_0(xt)J_0(ut)\,\mathrm{d}t, \quad 0 \leqslant x, u \leqslant 1$$

where $J_n(x)$ is the Bessel function of order n.

$$k(\varepsilon t) = -\Phi(\varepsilon t), \quad \lim_{t\to\infty} k(\varepsilon t) \neq 0 \quad \text{for } \varepsilon = \frac{a}{2\beta l} \neq 0$$

where l is the length of the crack.

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{4}\pi\right) \quad \text{for } x \gg 0$$

The limit of $tk(\varepsilon t)J_0(xt)J_0(ut)$ does not equal to zero for $t \to \infty$. So the kernel $L(x,u)$ in Eringen's paper is divergent (see e.g. Eringen (1979)). Of course, the triple integral equations can be considered to be a single integral equation of the first kind with a discontinuous kernel (see e.g. Eringen et al. (1977)). It is well known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. To overcome this difficulty, the Schmidt method (Morse and Feshbach, 1958) is used to solve the triple integral equations (27)–(30). The displacement w and the electric potential ϕ can be represented by the following series:

$$w(x,0) = \sum_{n=0}^{\infty} a_n P_n^{(1/2,1/2)}\left(\frac{x - \frac{1+b}{2}}{\frac{1-b}{2}}\right)\left(1 - \frac{\left(x - \frac{1+b}{2}\right)^2}{\left(\frac{1-b}{2}\right)^2}\right)^{1/2}, \quad \text{for } b \leqslant x \leqslant 1, \ y = 0 \tag{31}$$

$$w(x,0) = 0, \quad \text{for } x > 1, \ x < b, \ y = 0 \tag{32}$$

$$\phi(x,0) = \sum_{n=0}^{\infty} b_n P_n^{(1/2,1/2)}\left(\frac{x - \frac{1+b}{2}}{\frac{1-b}{2}}\right)\left(1 - \frac{\left(x - \frac{1+b}{2}\right)^2}{\left(\frac{1-b}{2}\right)^2}\right)^{1/2}, \quad \text{for } b \leqslant x \leqslant 1, \ y = 0 \tag{33}$$

$$\phi(x,0) = 0, \quad \text{for } x > 1, \ x < b, \ y = 0 \tag{34}$$

where a_n and b_n are unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial (Gradshteyn and Ryzhik, 1980). The Fourier transformation of Eqs. (31) and (33) is (Erdelyi, 1954)

$$A(s) = \bar{w}(s,0) = \sum_{n=0}^{\infty} a_n B_n G_n(s) \frac{1}{s} J_{n+1}\left(s\frac{1-b}{2}\right) \tag{35}$$

$$B(s) = \bar{\phi}(s,0) - \frac{e_{15}}{\varepsilon_{11}}\bar{w}(s,0) = \sum_{n=0}^{\infty}\left(b_n - \frac{e_{15}}{\varepsilon_{11}}a_n\right) B_n G_n(s) \frac{1}{s} J_{n+1}\left(s\frac{1-b}{2}\right) \tag{36}$$

$$B_n = 2\sqrt{\pi}\frac{\Gamma\left(n+1+\frac{1}{2}\right)}{n!} \tag{37}$$

$$G_n(s) = \begin{cases} (-1)^{n/2}\cos\left(s\frac{1+b}{2}\right), & n = 0,2,4,6,\ldots \\ (-1)^{(n+1)/2}\sin\left(s\frac{1+b}{2}\right), & n = 1,3,5,7,\ldots \end{cases} \tag{38}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eqs. (35) and (36) into Eqs. (27)–(30), Eqs. (28) and (30) can be automatically satisfied, respectively. Then the remaining Eqs. (27) and (29) reduce to the form:

$$\sum_{n=0}^{\infty} a_n B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) G_n(s) J_{n+1}\left(s\frac{1-b}{2}\right)\cos(sx)\,\mathrm{d}s = \frac{\pi}{2\mu}\tau_0(1+\lambda) \tag{39}$$

$$\sum_{n=0}^{\infty}\left(b_n-\frac{e_{15}}{\varepsilon_{11}}a_n\right)B_n\int_0^{\infty}\operatorname{erfc}(\varepsilon s)G_n(s)J_{n+1}\left(s\frac{1-b}{2}\right)\cos(sx)\,\mathrm{d}s=-\frac{\pi D_0}{2\varepsilon_{11}} \tag{40}$$

where $\lambda = e_{15}D_0/\varepsilon_{11}\tau_0$.

So we can obtain that $\mu a_n/(1+\lambda)$ does not depend on the materials constants. From Eqs. (39) and (40), it can be shown that the unknown coefficients a_n and b_n have relation as following:

$$b_n=\left(\frac{e_{15}}{\varepsilon_{11}}-\frac{D_0\mu}{\varepsilon_{11}T_0}\right)a_n, \quad T_0=\tau_0(1+\lambda)$$

For a large s, the integrands of Eqs. (39) and (40) almost decrease exponentially. So the semi-infinite integral in Eqs. (39) and (40) can be evaluated numerically by Filon's method (see e.g. Amemiya and Taguchi, 1969). Eqs. (39) and (40) can now be solved for the coefficients a_n by the Schmidt's method (Morse and Feshbach, 1958). For brevity, Eq. (39) can be rewritten as

$$\sum_{n=0}^{\infty}a_nE_n(x)=U(x), \quad b<x<1 \tag{41}$$

where $E_n(x)$ and $U(x)$ are known functions and a_n are unknown coefficients. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_b^1 P_m(x)P_n(x)\,\mathrm{d}x=N_n\delta_{mn}, \quad N_n=\int_b^1 P_n^2(x)\,\mathrm{d}x \tag{42}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x)=\sum_{i=0}^{n}\frac{M_{in}}{M_{nn}}E_i(x) \tag{43}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n=\begin{bmatrix} d_{00},d_{01},d_{02},\ldots,d_{0n}\\ d_{10},d_{11},d_{12},\ldots,d_{1n}\\ d_{20},d_{21},d_{22},\ldots,d_{2n}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ d_{n0},d_{n1},d_{n2},\ldots,d_{nn}\end{bmatrix}, \quad d_{ij}=\int_b^1 E_i(x)E_j(x)\,\mathrm{d}x \tag{44}$$

Using Eqs. (41)–(44), we obtain

$$a_n=\sum_{j=n}^{\infty}q_j\frac{M_{nj}}{M_{jj}} \tag{45}$$

with

$$q_j=\frac{1}{N_j}\int_b^1 U(x)P_j(x)\,\mathrm{d}x \tag{46}$$

5. Numerical calculations and discussion

From the references (see e.g. Itou, 1978, 1979; Zhou et al., 1999a,b), it can be seen that the Schmidt method is performed satisfactorily if the first 10 terms of infinite series to Eq. (41) are retained. The behavior of the maximum stress stays steady with the increasing number of terms in (41). Coefficients a_n and b_n are known, so that entire stress field and the electric displacement can be obtained. However, in fracture mechanics, it is important to determine stress τ_{yz} and the electric displacement D_y in the vicinity of the crack's tips. τ_{yz} and D_y along the crack line can be expressed respectively as

$$\begin{aligned}\tau_{yz}(x,0) &= -\frac{2}{\pi}\sum_{n=0}^{\infty}(c_{44}a_n + e_{15}b_n)B_n\int_0^{\infty}\operatorname{erfc}(\varepsilon s)G_n(s)J_{n+1}\left(s\frac{1-b}{2}\right)\cos(xs)\,\mathrm{d}s \\ &= -\frac{2\mu}{\pi(1+\lambda)}\sum_{n=0}^{\infty}a_nB_n\int_0^{\infty}\operatorname{erfc}(\varepsilon s)G_n(s)J_{n+1}\left(s\frac{1-b}{2}\right)\cos(xs)\,\mathrm{d}s \qquad (47)\end{aligned}$$

$$\begin{aligned}D_y(x,0) &= -\frac{2}{\pi}\sum_{n=0}^{\infty}(e_{15}a_n - \varepsilon_{11}b_n)B_n\int_0^{\infty}\operatorname{erfc}(\varepsilon s)G_n(s)J_{n+1}\left(s\frac{1-b}{2}\right)\cos(xs)\,\mathrm{d}s \\ &= -\frac{2D_0\mu}{\pi T_0}\sum_{n=0}^{\infty}a_nB_n\int_0^{\infty}\operatorname{erfc}(\varepsilon s)G_n(s)J_{n+1}\left(s\frac{1-b}{2}\right)\cos(xs)\,\mathrm{d}s = \frac{D_0}{\tau_0}\tau_{yz}(x,0) \qquad (48)\end{aligned}$$

For $\varepsilon = 0$ at $x = b$, 1, we have the classical stress singularity. However, so long as $\varepsilon \neq 0$, the semi-infinite integration and the series in Eqs. (47) and (48) are convergent for any variable x. Eqs. (47) and (48) give a finite stress all along $y = 0$, so there is no stress singularity at the crack tips. At $b < x < 1$, τ_{yz}/τ_0 and $D_y\tau_0/D_0$ are very close to unity, and for $x > 1$, τ_{yz}/τ_0 and $D_y\tau_0/D_0$ possess finite values diminishing from a finite value at $x = 1$ to zero at $x = \infty$. Since $\varepsilon/(1-b) > 1/100$ represents a crack length of less than 100 atomic distances (as stated by Eringen (1979)), we do not pursue solutions valid at such small crack sizes. From Eqs. (39) and (47), it can be obtained that the stress field does not depend on the material constants except the lattice parameter and the crack length. So in all computation, the material constants are not considered. Here, we just give the stress field in this paper. The electric displacement field can be obtained by the stress field using Eq. (48). The results are plotted in Figs. 2–6. The following observations are very significant:

(i) The maximum stress does not occur at the crack tip, but slightly away from it. This phenomenon has been thoroughly substantiated by Eringen (1983). The maximum stress is finite. The distance between the crack tip and the maximum stress point is very small, and it depends on the crack length and the lattice parameter. It is difficult to determine the distance from the computation. For the sake of analytical simplification and the consistency with the local fracture theory, we just give the stress concentration value at the crack tips. It does not mean that the stress concentration occurs at the crack tips. The stress concentration values at the crack tip (as stated by Eringen (1978, 1979)) can be given by

$$\tau_{yz}(1,0)/\tau_0 = c_{\mathrm{R}}/\sqrt{a/[2\beta(1-b)]} \qquad (49)$$

$$\tau_{yz}(b,0)/\tau_0 = c_{\mathrm{L}}/\sqrt{a/[2\beta(1-b)]} \qquad (50)$$

where c_{R}, and c_{L} represent the stress concentration value at right tip and at left tip for right crack, respectively. The c_{R} is about equal to $c_{\mathrm{R}} \approx 0.2708$, 0.27275, 0.2764, 0.2787, 0.28071 for $b = 0.5$, 0.4, 0.3, 0.2 and 0.1, respectively. The c_{L} is about equal to 0.3794, 0.38843, 0.4021, 0.4242 and 0.50821 for $b = 0.5$, 0.4, 0.3, 0.2 and 0.1, respectively. c_{R} and c_{L} decrease with the increasing of the distance between two cracks, but the c_{R} changes slowly.

(ii) The stress at the crack tip becomes infinite as the atomic distance $a \to 0$. This is the classical continuum limit of square root singularity. This can be shown from Eqs. (27)–(30). For $a \to 0$, Eqs. (27)–(30) will reduce to the triple integral equations for the same problem in classical piezoelectric materials. Contrary to the classical piezoelectric theory solution, it is found that no stress and electric displacement sin-

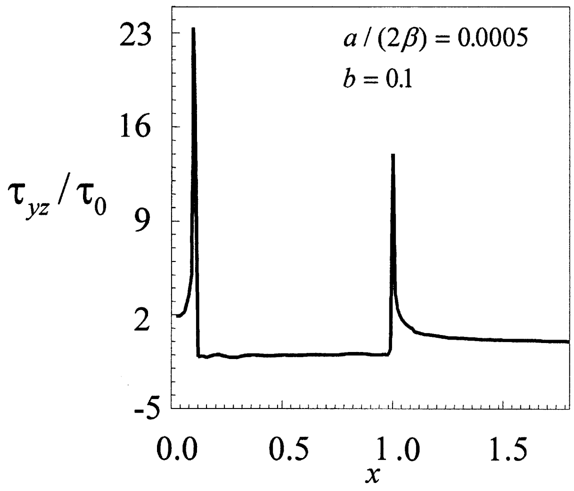

Fig. 2. The variation of anti-plane shear stress along the crack line.

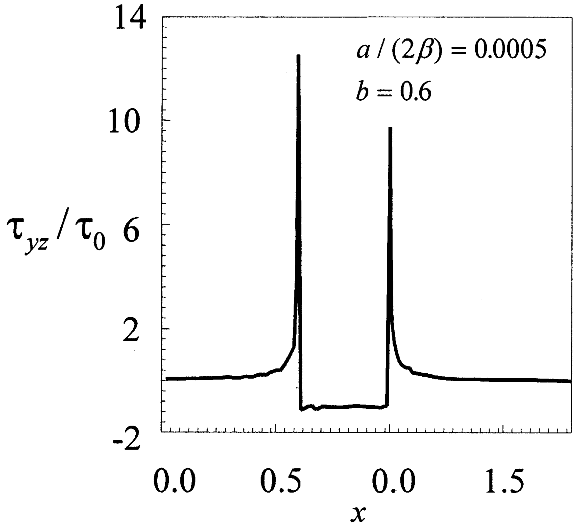

Fig. 3. The variation of anti-plane shear stress along the crack line.

gularity at the crack tips, and also the present results converge to the classical ones when far away from the crack tip.

(iii) For $a/\beta =$ constant, i.e., the atomic distance does not change, the value of the stress concentrations values (at the crack tip) increase with the increase of the crack length ($a/2\beta l$ will becomes smaller with the increase of the crack length l). Noting this fact, experiments indicate that the piezoelectric materials with smaller cracks are more resistant to fracture than those with larger cracks.

(iv) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales, viz., it may solve the problem of any scale cracks (it may solve the problem of any value of $a/2\beta l$).

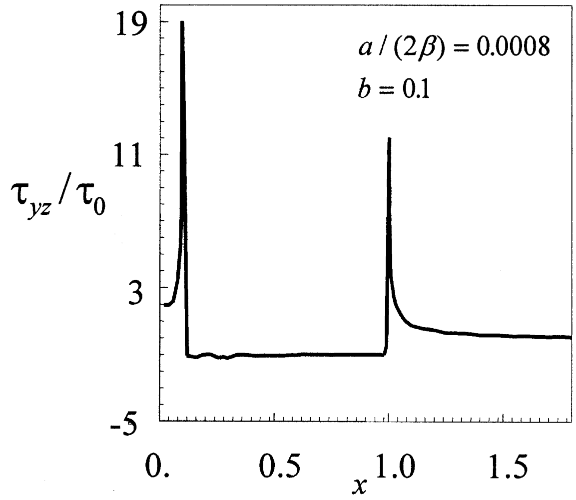

Fig. 4. The variation of anti-plane shear stress along the crack line.

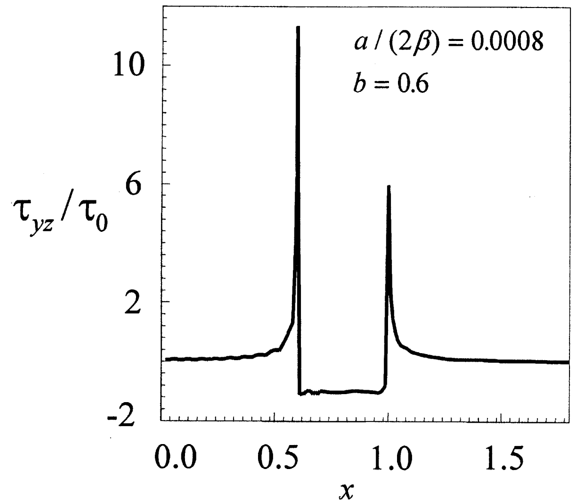

Fig. 5. The variation of anti-plane shear stress along the crack line.

(v) The left tip's stress is greater than the right tip's for the right crack. At the left end of the right crack, the stress on the crack line decreases with the increasing of the distance between two cracks.

(vi) The dimensionless stress field is found to be independent of the electric loads D_0 and the material parameters. They just depend on the length of the crack $1 - b$ and the lattice parameter a. The stress field is not coupled with the electric field. This is consistent with the piezoelectric theory for the impermeable crack surface conditions in the piezoelectric materials plane. The electric displacement field just depends on the stress loading τ_0, the electric loading D_0, the length of the crack $1 - b$, the lattice parameter a. However, for permeable case, the electric displacement field is independent of ε_{11}. It depends on the stress loading τ_0, the

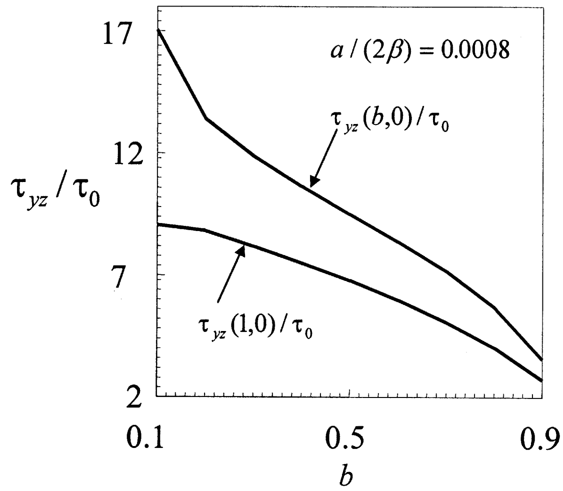

Fig. 6. The variation with b of anti-plane shear stress at the crack tips.

length of the crack $1-b$, the lattice parameter a, the shear modulus c_{44} and piezoelectric coefficient e_{15} (Zhou et al., submitted for publication).

(vii) In contrast to the permeable crack surface condition solution, it is found that the electric displacement for the impermeable crack surface conditions is much larger than the results for the permeable crack surface conditions (see, Zhou et al., submitted for publication).

(viii) The stress field $\tau_{yz}(x,0)$ has a relation to the electric displacement field $D_y(x,0)$ along the crack line in the form of $D_y(x,0) = (D_0/\tau_0)\tau_{yx}(x,0)$. Whereas, for permeable crack boundary conditions, the relation is $D_y(x,0) = (e_{15}/c_{44})\tau_{yx}(x,0)$ along the crack line.

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